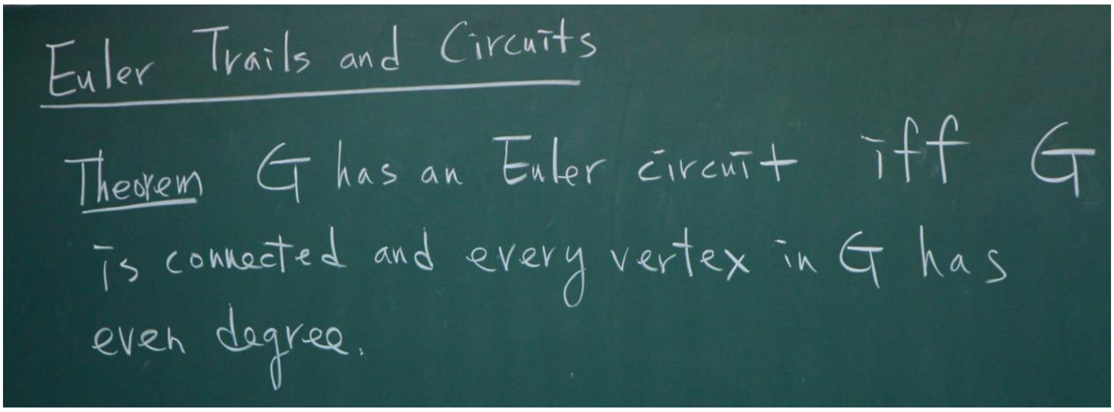
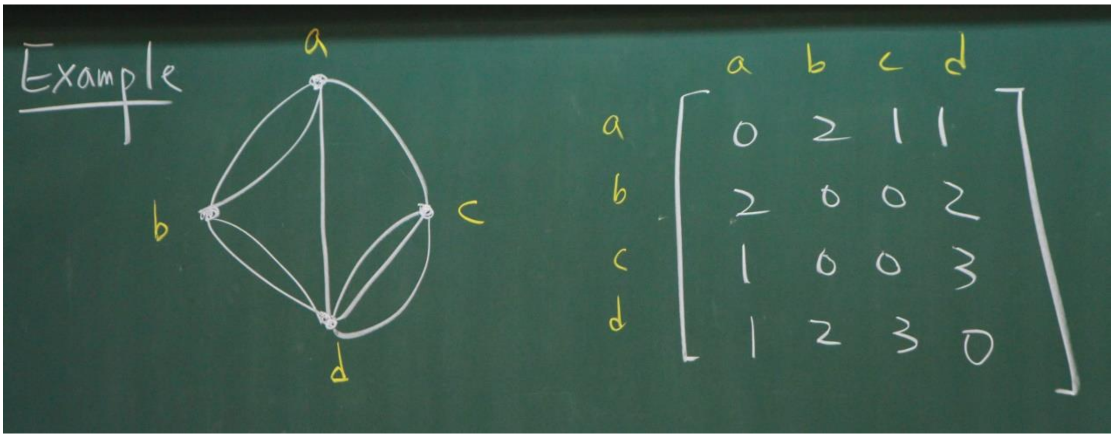
【10920 趙啟超教授離散數學 / 第 21 堂版書】

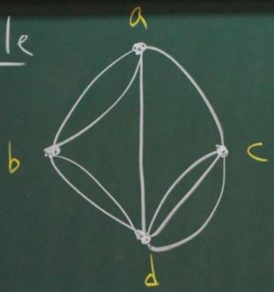




Euler Trails and Circuits

Theorem G has an Euler circuit iff G is connected and every vertex in G has even degree.

Example



$$
\begin{array}{c|cccc}
 & a & b & c & d \\
\hline
a & 0 & 2 & 1 & 1 \\
b & 2 & 0 & 0 & 2 \\
c & 1 & 0 & 0 & 3 \\
d & 1 & 2 & 3 & 0
\end{array}
$$

$$
\begin{array}{c|cccc}
 & a & b & c & d \\
\hline
a & 0 & 0 & 0 & 0 \\
b & 0 & 0 & 0 & 0 \\
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d & 0 & 0 & 2 & 0
\end{array}
$$

a b a c d a

b a c d a b (break out at b)

b a c d a b d b

c d a b d b a c (break out at c)

$$\begin{array}{c|cccc} b & 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 2 \\ d & 0 & 0 & 2 & 0 \end{array}$$

b a c d a b (break out at b)

b a c d a b d b

c d a b d b a c (break out at c)

$$\begin{array}{c|c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

c d a b d b a c d c (Euler circuit)

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a b a c d a

b a c d a b (break out at b)

b a c d a b d b

c d a b d b a c (break out at c)

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c d a b d b a c d c (Euler circuit)

<u>Corollary</u> G has an Euler trail iff G is connected and has exactly two vertices of odd degree.

<u>Proof</u> "$\Rightarrow$" The proof is almost the same as the proof for the first part of the theorem for Euler circuit. The only difference is that the beginning and

ending vertices have odd degree.

"$\Leftarrow$" Add an imaginary edge connecting the two vertices of odd degree. Find an Euler circuit in the new graph, starting with the imaginary edge. Then delete the imaginary edge from the Euler circuit. We thus obtain an Euler trail. ∎

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Planar Graphs

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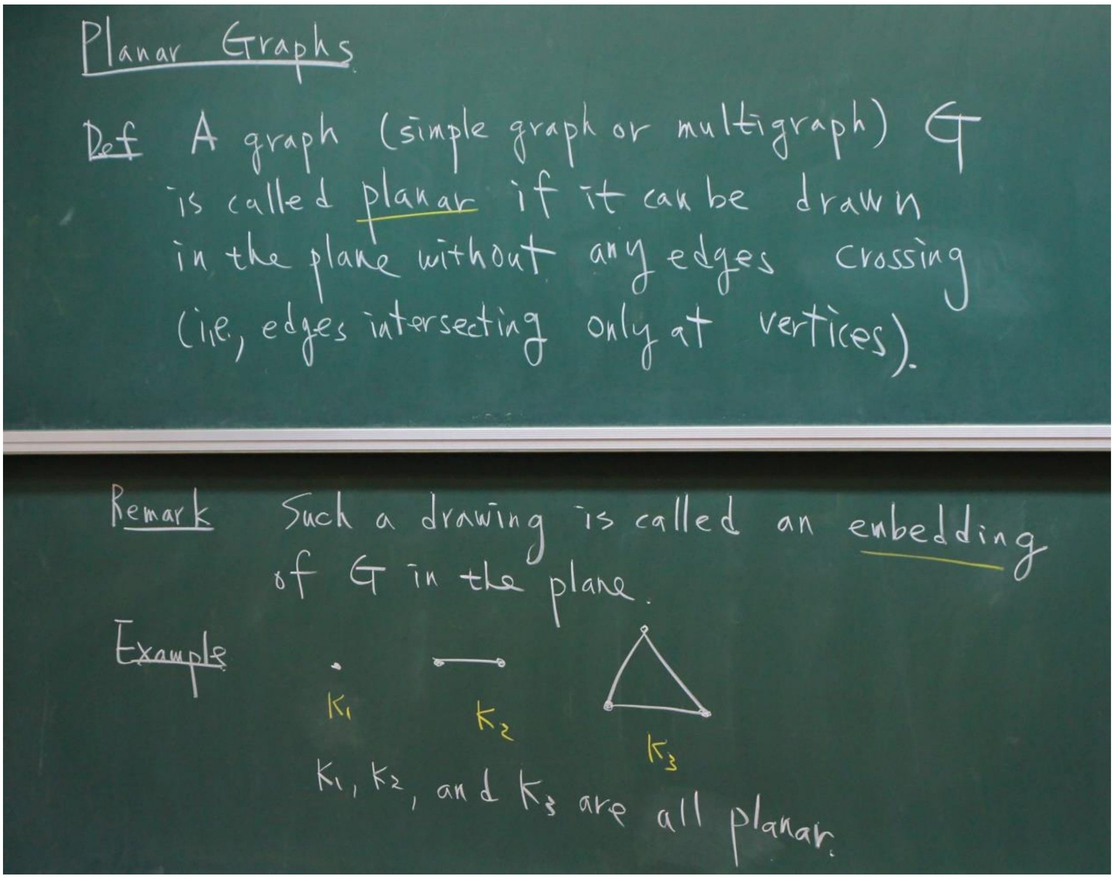
Remark Such a drawing is called an embedding of G in the plane.

Example

K_1 K_2 K_3

K_1, K_2, and K_3 are all planar.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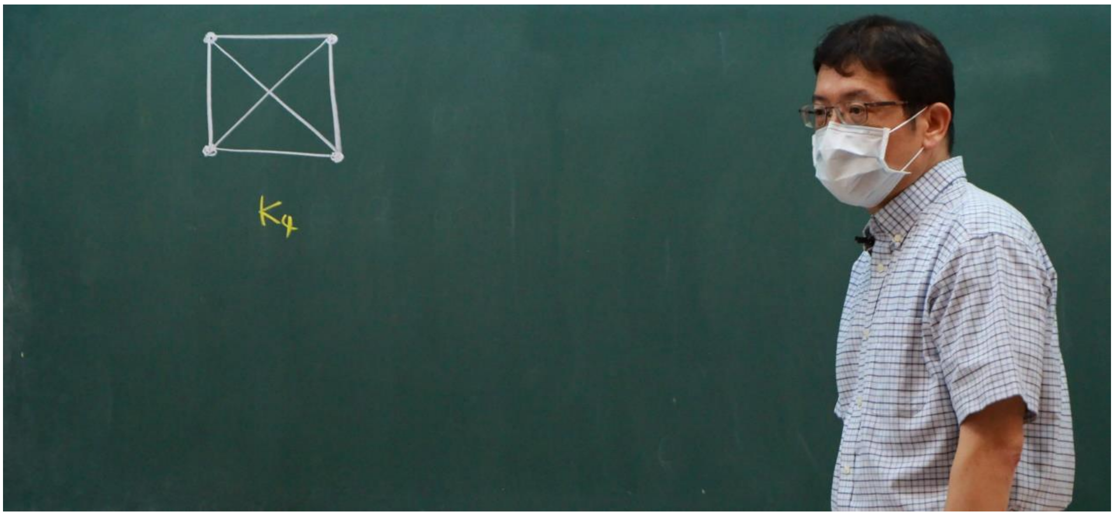
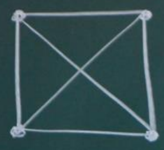
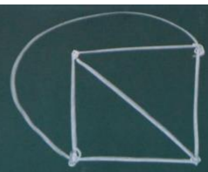

K_4

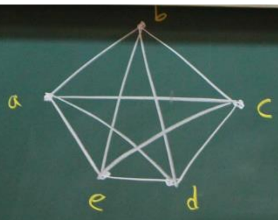


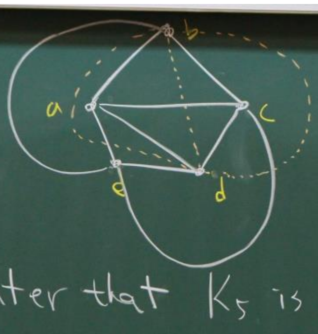
K_4



K_4 is also planar.

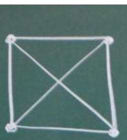
<u>Example</u> Is K_5 planar?





Where to put the edge $\{b, d\}$?

We will prove later that K_5 is nonplanar.



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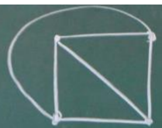

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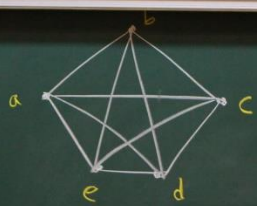

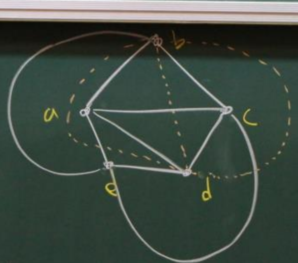

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Def A simple graph G is called bipartite if its vertex set can be partitioned into two disjoint sets V_1 and V_2 such that every edge connects a vertex in V_1 and a vertex in V_2.

Example Complete bipartite graph $K_{m,n}$

Each vertex in V_1 is joined with every vertex in V_2 with $|V_1|=m$ and $|V_2|=n$.

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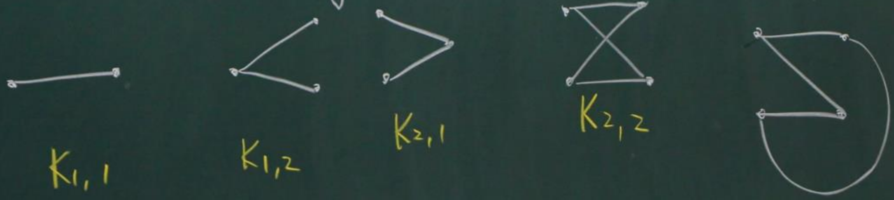
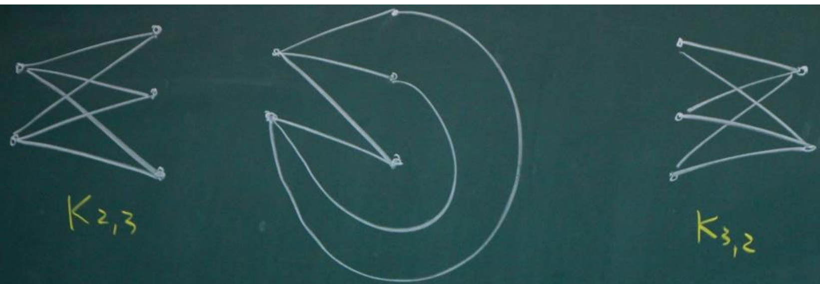


$K_{1,1}$ $K_{1,2}$ $K_{2,1}$ $K_{2,2}$



$K_{2,3}$ $K_{3,2}$

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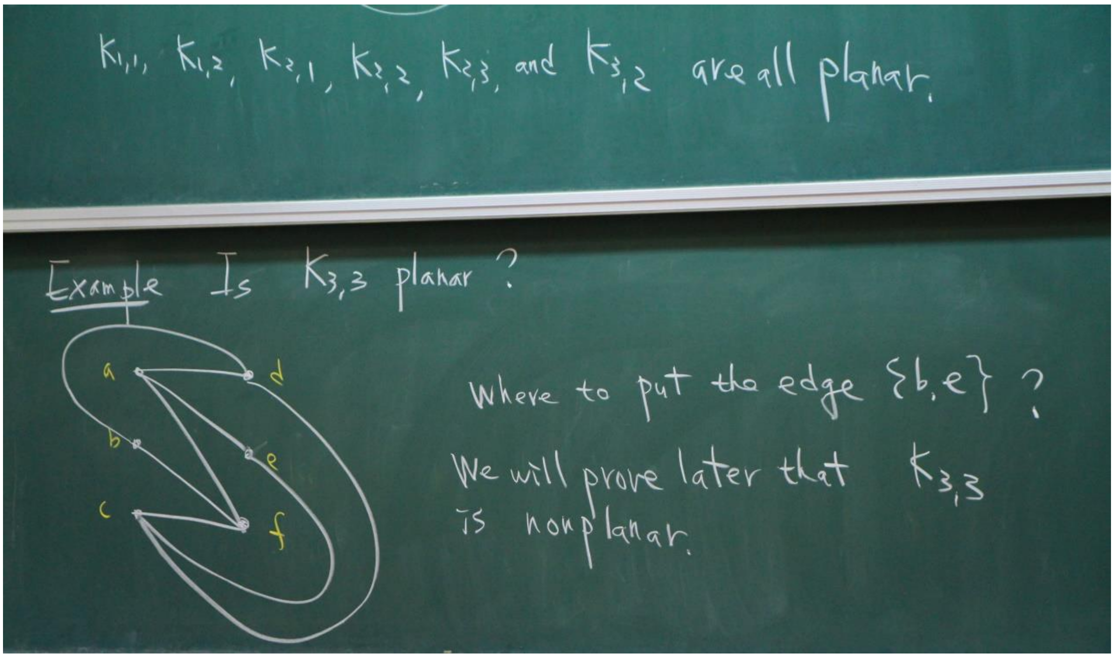

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a
b
c
d
e
f
Where to put the edge $\{b, e\}$?
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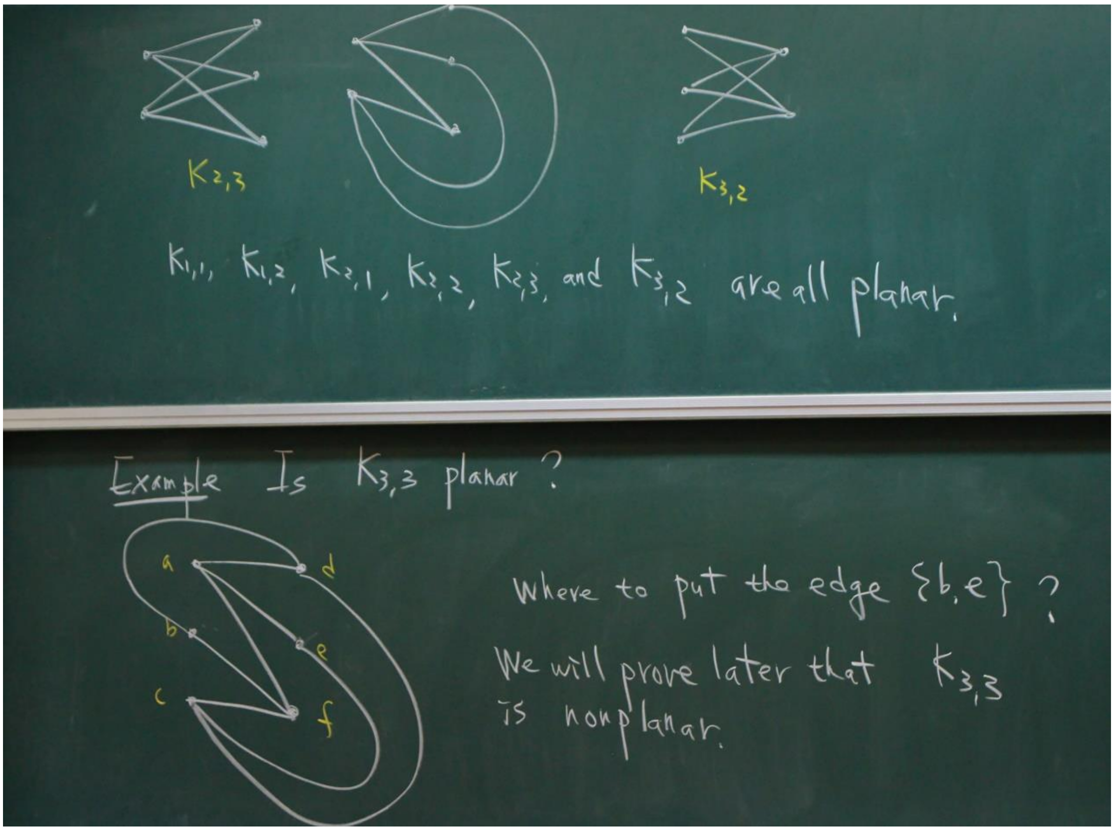

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An embedding of a planar graph splits the plane into regions.

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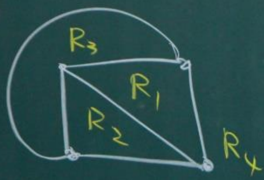

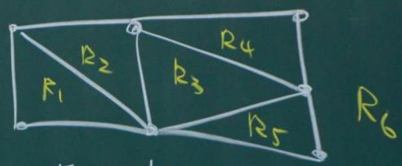

4 vertices
6 edges
4 regions

7 vertices
11 edges
6 regions

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$4-6+4=2$ $\quad$ $7-11+6=2$

Suppose $G=(V,E)$ is a connected planar graph.

Let $|V|=v$, $|E|=e$, and there be r regions.

Theorem (Euler's Theorem for Planar Graphs)

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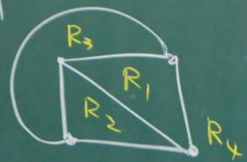

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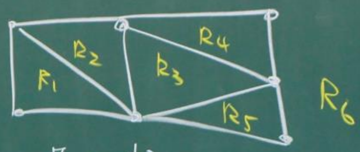

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Proof The proof is by induction on e.

If $e=0$ or 1, then G is isomorphic to one of the graphs in (a), (b), or (c).

(a) $\qquad$ (b) $\qquad$ (c)

(a) $v-e+r = 1-0+1=2$

(b) $v-e+r = 1-1+2 = 2$

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In all three cases, $v-e+r=2$.

Assume that the result is true for every connected planar graph with e edges, where $0 \le e \le k$.

If $G=(V,E)$ is a connected planar graph with v vertices, r regions, and $e=k+1$ edges, let $a,b \in V$ with $\{a,b\} \in E$.

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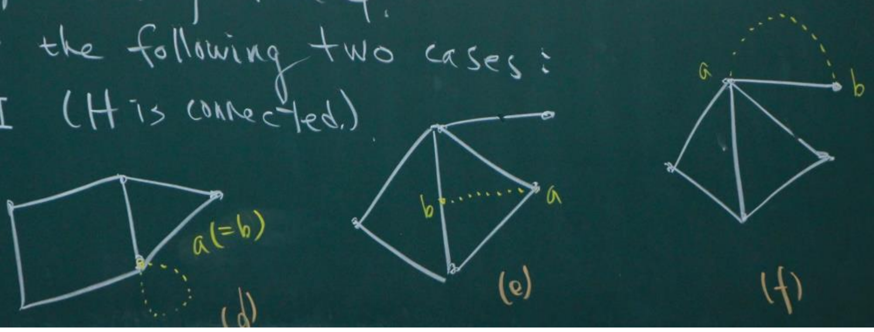
We will show that the result is true for G.

Consider the graph H obtained by deleting the edge $\{a, b\}$ from G.

Consider the following two cases:

Case I (H is connected.)



In all these situations ((d), (e), (f)), H has v vertices, $e-1 = k$ edges, and $r-1$ regions.

The induction hypothesis applied to H gives that

$$v - k + (r-1) = 2,$$ from which it follows that

$$2 = v-(k+1)+r = v-e+r.$$

So Euler's Theorem is true for G in this case.

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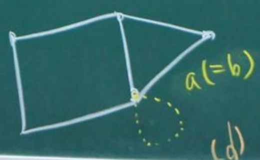
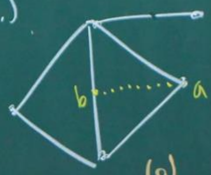
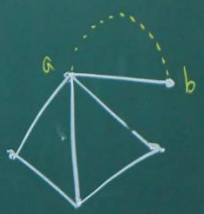


(d)



(e)



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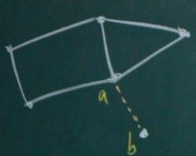
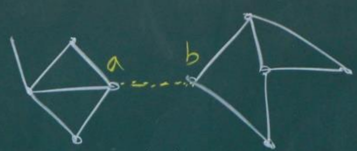


(g)



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Here H has two components H_1 and H_2, where H_i has v_i vertices, e_i edges, and r_i regions, for $i = 1, 2$.

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Consequently, $(v_1+v_2)-(e_1+e_2)+(r_1+r_2)$
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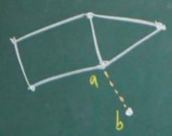

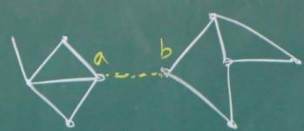

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Remark If the graph has no cycles, we define its girth to be the total number of edges in the graph.

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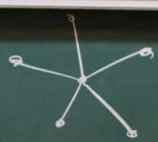
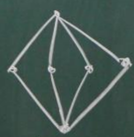


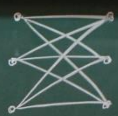
girth $g = 5$



$g = 4$

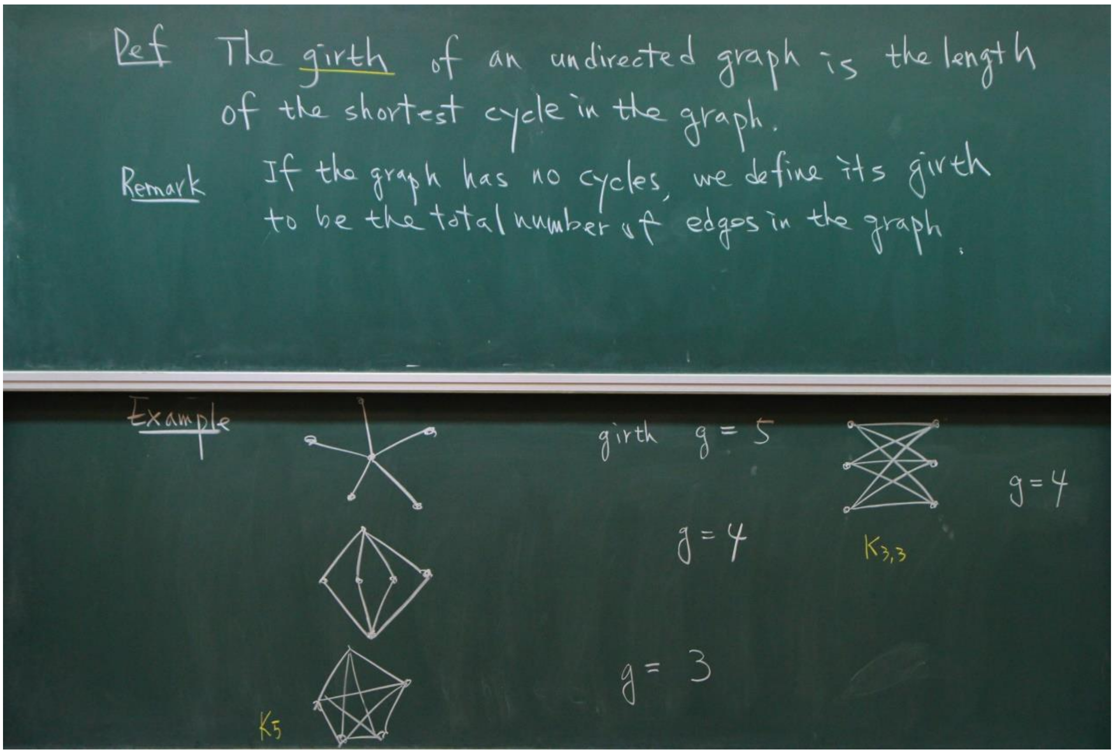
K_5

$g = 3$



$K_{3,3}$

$g = 4$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Remark If the graph has no cycles, we define its girth to be the total number of edges in the graph.
Example
girth g = 5
g = 4
g = 3
K5
g = 4
K3,3